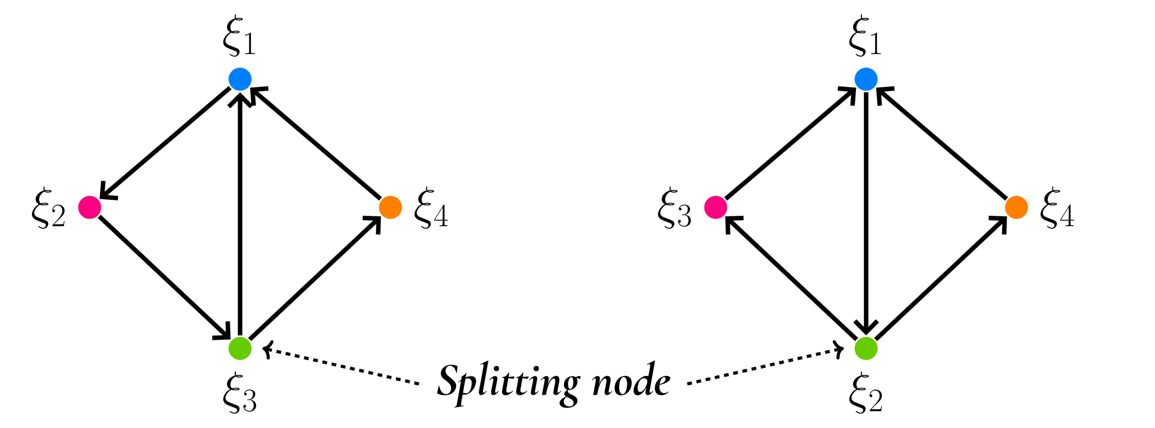
Switching near heteroclinic networks as a piecewise-smooth dynamical system

David Groothuizen Dijkema

Heteroclinic networks

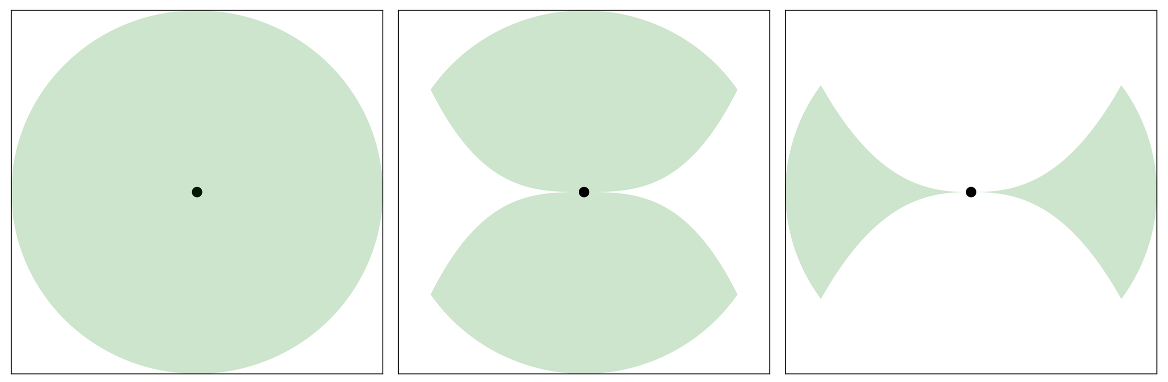
In a dynamical system, a ***heteroclinic cycle*** is an invariant set of equilibria and connecting heteroclinic orbits. A ***heteroclinic network*** is a connected union of heteroclinic cycles. Two of the three possible heteroclinic networks in $\mathbb{R}^4$ are represented below.



The Δ-clique Network   The Kirk-Silber Network [2]

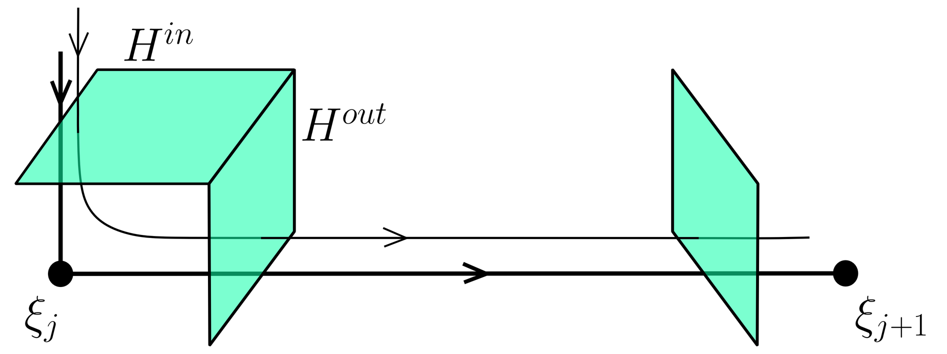
When analysing heteroclinic networks, we are often interested in questions such as:

i) When are the network and its component cycles stable, and how much so?
ii) How do trajectories evolve near the network? Can there be switching between cycles, how often, and in which direction?



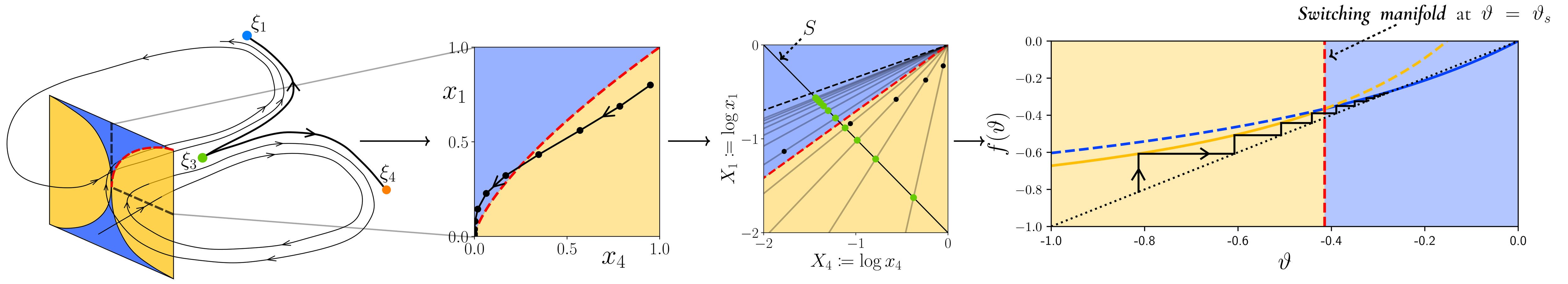
Asymptotic stability   Essential asymptotic stability   Fragmentary asymptotic stability

The black dots above represent an invariant set, such as a heteroclinic network or cycle, and we shade the basin of attraction green. ***Essential*** and ***fragmentary asymptotic stability*** (e.a.s. and f.a.s., respectively) are two ways a heteroclinic cycle can be attracting, but not asymptotically stable.



To analyse heteroclinic networks, we produce ***return maps*** approximating the flow near the network by linearising the flow near equilibria and along heteroclinic orbits. Stability properties can be derived from these maps using ***stability indices*** [4].

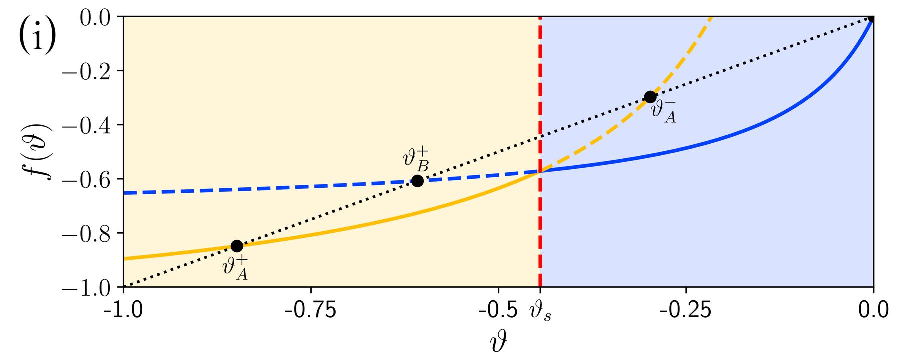
Analysing the Δ-clique network



Construct a Poincaré section at the splitting node. Construct a return map on the section. Take logarithms and project onto a unit simplex, S. Construct a continuous, piecewise-smooth ***projected map*** f on S induced by the action of the return map in log coordinates.

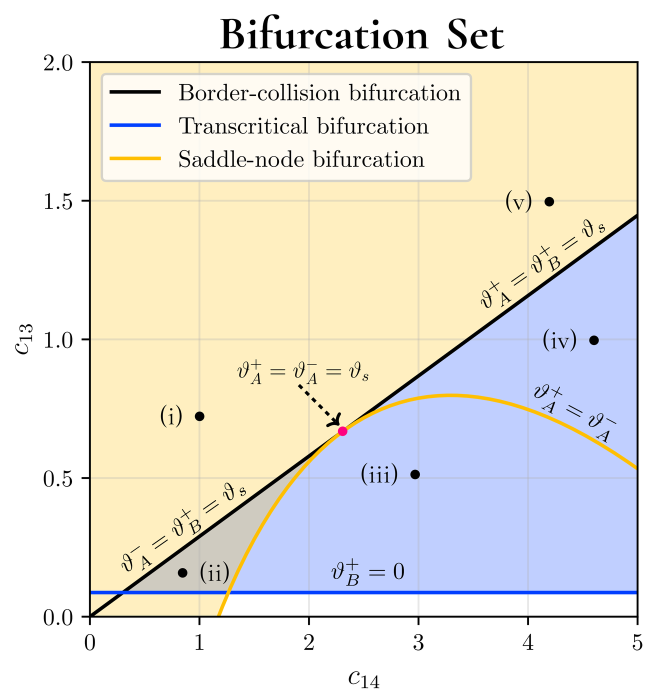
Stability and switching in the Δ-clique network

(i)

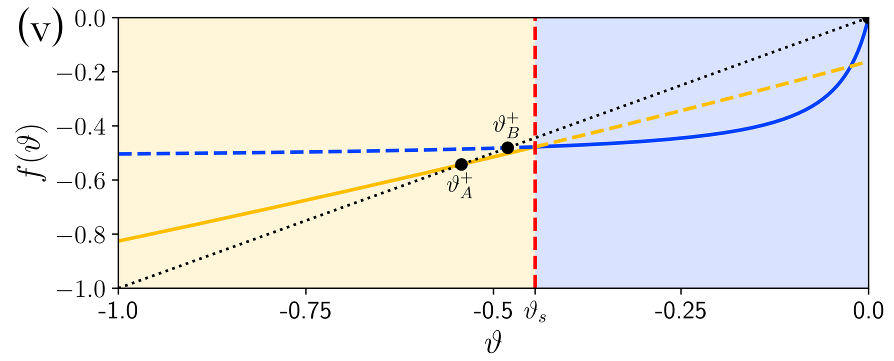


$\mathcal{C}_A$ is f.a.s. and $\mathcal{C}_B$ is c.u. Trajectories switch from $\mathcal{C}_B$ to $\mathcal{C}_A$.

ϑ_A^- and ϑ_B^+ cross the switching manifold simultaneously in opposite directions.



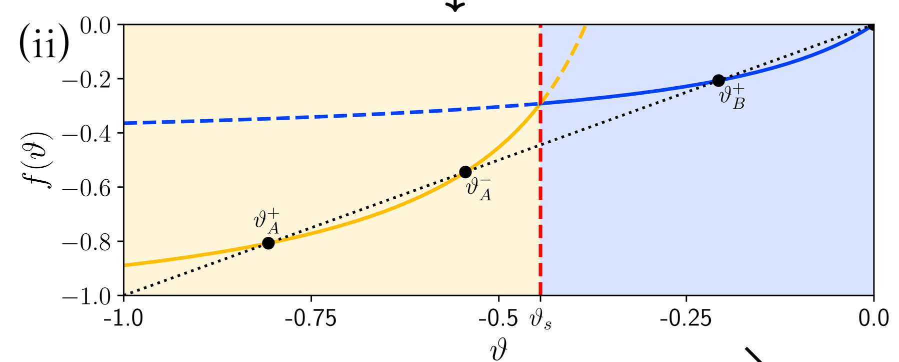
(v)



$\mathcal{C}_A$ is f.a.s. and $\mathcal{C}_B$ c.u. Trajectories switch from $\mathcal{C}_B$ to $\mathcal{C}_A$.

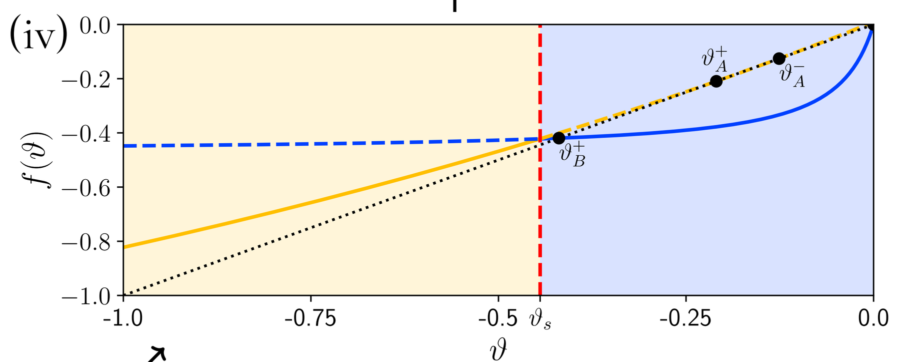
ϑ_A^+ and ϑ_B^+ cross the switching manifold simultaneously in the same direction.

(ii)



$\mathcal{C}_A$ and $\mathcal{C}_B$ are both f.a.s. Some trajectories switch from $\mathcal{C}_A$ to $\mathcal{C}_B$.

(iv)

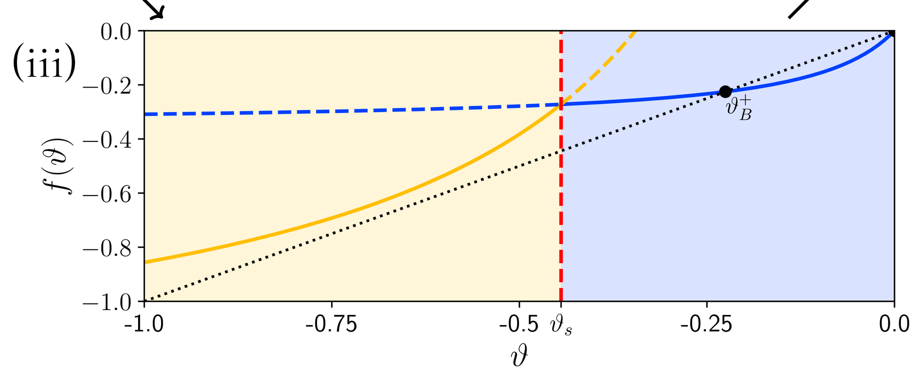


$\mathcal{C}_B$ is f.a.s. and $\mathcal{C}_A$ c.u. Trajectories switch from $\mathcal{C}_A$ to $\mathcal{C}_B$.

ϑ_A^+ and ϑ_A^- are destroyed in an ***admissible*** saddle-node bifurcation.

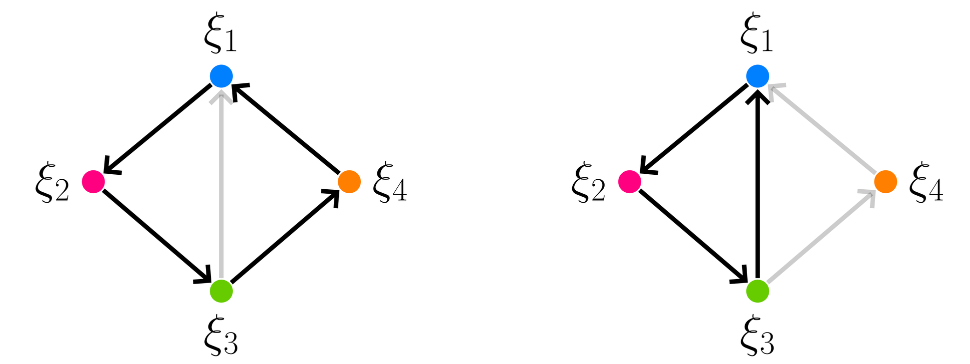
ϑ_A^+ and ϑ_A^- are created in a ***virtual*** saddle-node bifurcation.

(iii)



$\mathcal{C}_B$ is f.a.s. and $\mathcal{C}_A$ c.u. Trajectories switch from $\mathcal{C}_A$ to $\mathcal{C}_B$.

- $\mathcal{C}_A$ - $\xi_1 \to \xi_2 \to \xi_3 \to \xi_4 \to \xi_1 \ldots$
- $\mathcal{C}_B$ - $\xi_1 \to \xi_2 \to \xi_3 \to \xi_1 \ldots$
- $\vartheta_A^\pm$ - fixed points of f_A; ϑ_A^+ corresponds to $\mathcal{C}_A$
- $\vartheta_B^\pm$ - fixed points of f_B; ϑ_B^+ corresponds to $\mathcal{C}_B$
- c.u. - completely unstable
- f.a.s. - fragmentarily asymptotically stable



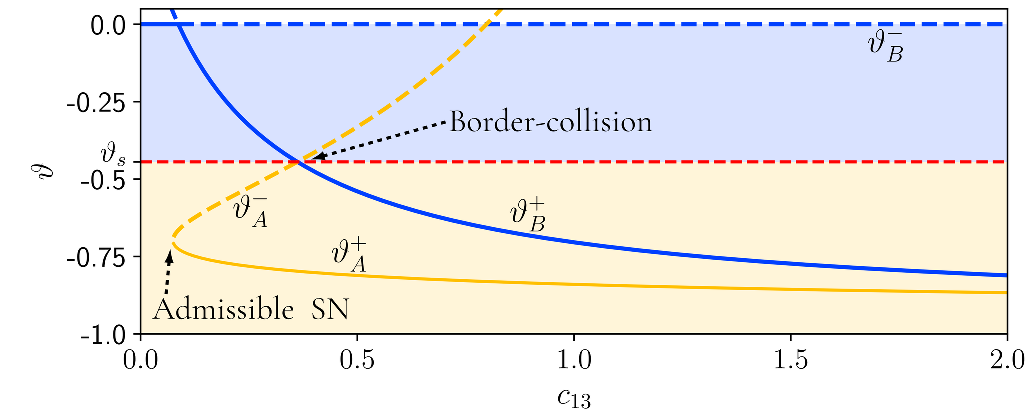
The $\mathcal{C}_A$ cycle.   The $\mathcal{C}_B$ cycle.

Bifurcation Diagrams

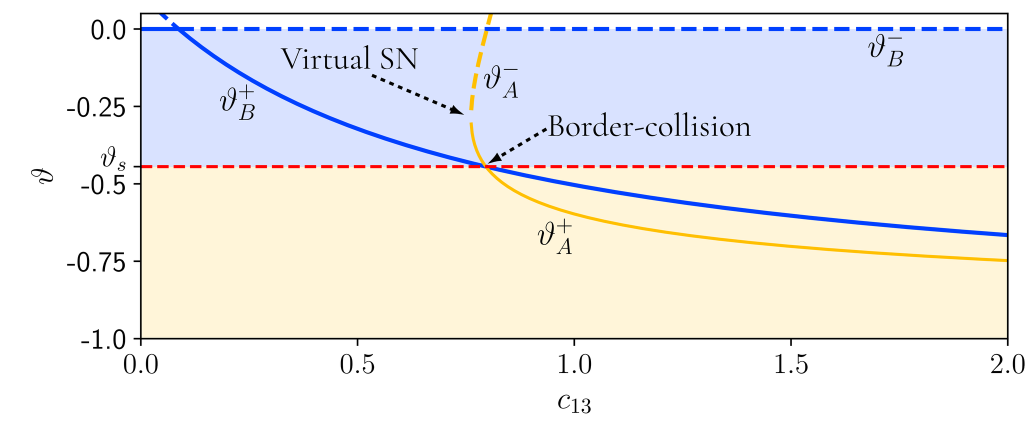
The map $f\colon S \to S$ is a piecewise map of two components, $f_A\colon \Theta_A \to S$ and $f_B\colon \Theta_B \to S$, where $\Theta_A = (-1, \vartheta_s)$ and $\Theta_B = (\vartheta_s, 0)$.

If a fixed point ϑ_A of f_A lies in Θ_A, it is ***admissible***. If it lies in Θ_B it is ***virtual***. The point $\vartheta_A = \vartheta_s$ is known as a ***border-collision bifurcation*** [1]. (And likewise for f_B.)

Below we show two bifurcation diagrams of the fixed points of f against c_{13} for fixed values of c_{14} (the eigenvalues at ξ_1 in the x_3 and x_4 directions). In both diagrams, stable fixed points are solid lines and unstable fixed point are dashed lines. The diagrams are coloured amber and blue in the domain of definition of f_A and f_B, as are the fixed points of each function.



Bifurcation diagram for $c_{14} = 1.25$: ϑ_A^- and ϑ_B^+ simultaneously cross ϑ_s as c_{13} increases and both lose admissibility.



Bifurcation diagram for $c_{14} = 2.75$: ϑ_A^+ and ϑ_B^+ simultaneously cross ϑ_s as c_{13} increases and exchange admissibility.

Future Work

- Analyse sustained switching near a heteroclinic network (such as the Rock-Paper-Scissors-Lizard-Spock network [5] or the two cycle network of Podvigina [3]) as periodic orbits in the continuous, piecewise smooth projected map.
- Analyse the "strings of sausages" stability regions identified in [5] as border-collision bifurcations of these periodic orbits.

References

[1] M. Bernardo, C. J. Budd, A. R. Champneys, and P. Kowalczyk. *Piecewise-smooth Dynamical Systems: Theory and Applications*. Applied Mathematical Sciences. Springer London, 2008.

[2] V. Kirk and M. Silber. A competition between heteroclinic cycles. *Nonlinearity*, 7:1605–1621, 1994.

[3] O. Podvigina. Behaviour of trajectories near a two-cycle heteroclinic network. *arXiv preprint arXiv:2107.09982*, 2021.

[4] O. Podvigina and P. Ashwin. On local attraction properties and a stability index for heteroclinic connections. *Nonlinearity*, 24:887–929, 2011.

[5] C. M. Postlethwaite and A. M. Rucklidge. Stability of cycling behaviour near a heteroclinic network model of Rock–Paper–Scissors–Lizard–Spock. *Nonlinearity*, 35:1702–1733, 2022.